MATH 2850: THE DIRAC DELTA 'FUNCTION'

DEFINITION: Given a, $h > 0$, define $\delta_h(t-a) = \begin{cases} \dfrac{1}{2h}, & a-h \le t \le a+h \\ 0, & \text{otherwise.} \end{cases}$

EXAMPLE: Graph $\delta_2(t-3)$, $\delta_1(t-3)$, $\delta_{\frac{1}{2}}(t-3)$, and $\delta_{\frac{1}{4}}(t-3)$ on the same axes.

QUESTION: What appears to be $\lim\limits_{h \to 0^+} \delta_h(t-3)$? Prove your answer using the definition of $\delta_h(t-3)$.

EXAMPLE: Find $\lim\limits_{h \to 0^+} \delta_h(t-a)$.

DEFINITION: The 'function' $\delta(t-a) = \lim\limits_{h \to 0^+} \delta_h(t-a)$ is called the Dirac Delta 'function'.

Usually, anything involving $\delta(t-a)$ is replaced by $\lim\limits_{h \to 0^+} \delta_h(t-a)$.

Physically, the Dirac Delta 'function' represents an 'instantaneous impulse' like plucking a string.

Loosely, $\delta(t-a) = D_t\left[\mathcal{U}(t-a)\right]$ since $\mathcal{U}$ needs to climb from $y = 0$ to $y = 1$ instantaneously at $t = a$.

THEOREM: (Sifting Property): $\displaystyle\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)$

NOTE: $\delta(-u) = \delta(u)$, so $\delta(t-a) = \delta(a-t)$. Hence, $\displaystyle\int_0^\infty f(t)\,\delta(t-a)\,dt = \int_0^\infty f(t)\,\delta(a-t)\,dt$.

Relabeling, we get: $f(t) = \displaystyle\int_0^\infty f(u)\,\delta(t-u)\,dt = f(t) * \delta(t)$. That is, f is a sum of impulse responses.

THEOREM: $\mathcal{L}\{\delta(t-a)\} = e^{-as}$. In particular, $\mathcal{L}\{\delta(t)\} = 1$

PROOF:

EXAMPLE: Solve the IVP: $y'' + 16y = \delta(t-2\pi)$; $y(0) = y'(0) = 0$.

Graph the solution. What happens at $t = 2\pi$?

Ans: $y(t) = \dfrac{1}{4}\,\mathcal{U}(t-2\pi)\,\sin(4t)$

RECALL: Solve the following IVP in terms of convolutions: $ay'' + by' + cy = f(t)$, $y(0) = k_0$ and $y'(0) = k_1$.

To get the 'impulse' response, we set set $f(t) = \delta(t)$ and get $w(t) = \mathcal{L}^{-1}\{1/p(s)\}$ from last time.

HOMEWORK: Section 8.7: Pg. 461: 1 - 25 odd